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INTEGER-VALUED FUNCTIONS AND INCREASING UNIONS OF FIRST COUNTABLE SPACES†

BY

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ABSTRACT

We determine those regular cardinals κ with the property that for each increasing κ-chain of first countable spaces there is a compatible first countable topology on the union of the chain. Assuming $V = L$ any such κ must be weakly compact. It is relatively consistent with a supercompact cardinal that each $\kappa > \omega_1$ has the property. The proofs exploit the connection with interesting families of integer-valued functions.

A natural and interesting investigation in set-theoretic topology is to determine the cardinal invariants of a union of a chain of spaces, based on a knowledge of the invariants for each of the spaces in the chain. When we say "increasing chain of topological spaces", we mean that if $\alpha < \alpha'$ then X_α is a subspace of $X_{\alpha'}$. Many theorems of this nature appear in [3]. Here, we are interested in the character of a space which is the union of a chain of first countable spaces. In [3] there is a proof of the following theorem. If $\kappa \geqq \aleph_2$ is a regular cardinal and $\{X_\alpha : \alpha < \kappa\}$ is an increasing chain of first countable spaces, then any compact Hausdorff topology on the union, such that each X_α is a subspace, is also first countable. This compactness condition, however, is very restrictive. Indeed, given an increasing chain of first countable spaces there may be no compact Hausdorff topology on the union which "extends" the chain, i.e. has each member of the chain as a subspace.

We answer the question of which κ have the property that there is a

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compatible first countable topology for the union of each increasing chain $\{X_\alpha : \alpha < \kappa\}$ of first countable spaces. This question for κ, the successor of $2^{\aleph_0}$, was brought to our attention by Andrew Berner. He observed that from an affirmative answer one could obtain a very easy proof of Theorem 1 in [4]. It is easy to see that without loss of generality we need only consider chains with length equal to some regular cardinal κ. We call these κ-chains.

There are two natural topologies to put on the union of a κ-chain.

DEFINITION 1. Suppose that X is the union of a κ-chain $\{X_\alpha : \alpha < \kappa\}$. The *fine* topology is the largest, or finest, topology which can be put on X such that each X_α is a subspace of X. The *weak* topology is the smallest, or coarsest, topology which can be put on X such that each X_α is a subspace of X.

The reader can easily check that a set U is open in the fine topology if and only if $U \cap X_\alpha$ is open for each $\alpha < \kappa$. Similarly the family

$$\left\{X - \bigcup_{\beta \leqq \alpha < \kappa} \mathrm{cl}_{X_\alpha} F : F \subset X_\beta\right\}$$

forms a base for the weak topology on X.

It is immediate that the weak topology on an $\aleph_0$-chain of first countable spaces is again first countable. Also, in almost all cases the fine topology on an $\aleph_0$-chain is *not* first countable. Surprisingly for $\kappa > \aleph_0$ there is at *most* one compatible first countable topology for each κ-chain.

THEOREM 2. *Suppose that κ is a regular uncountable cardinal and that $\{X_\alpha : \alpha < \kappa\}$ is a κ-chain of first countable spaces. Suppose also that the union X is endowed with a compatible topology so that $p \in X$. If $\chi(p, X) < \kappa$ then $\chi(p, X) = \omega$ and the topology at p is the fine topology.*

PROOF. Let $\mathscr{W}$ be a local base at p with $|\mathscr{W}| < \kappa$. Fix $\beta < \kappa$ large enough so that if V, $W \in \mathscr{W}$ are such that $V - W \neq \varnothing$ then $V - W \cap X_\beta \neq \varnothing$. Since X_β is first countable we can choose a countable $\mathscr{V} \subset \mathscr{W}$ so that $\{V \cap X_\beta : V \in \mathscr{V}\}$ is a base for p in X_β. But this means that $\mathscr{V}$ is a local base for p in X. Now suppose that $U \subset X$ is a neighbourhood of p in the fine topology on X. For each $V \in \mathscr{V}$ choose, if possible, $\alpha_V < \kappa$ so that $V - U \cap X_{\alpha_V} \neq \varnothing$. Choose $\alpha < \kappa$ so that $\alpha_V < \alpha$ for each $V \in \mathscr{V}$. Let W be an X_α-neighbourhood of p such that $W \subset U$. Now choose a $V \in \mathscr{V}$ so that $V \cap X_\alpha \subset W$, therefore it must be the case that $V \subset U$ and $\mathscr{V}$ is a base at p in the fine topology. ■

We can now consider the following question.

MAIN QUESTION. Suppose that κ is a regular cardinal and X, with the fine topology, is the union of an increasing κ-chain of first countable spaces. Is X necessarily first countable?

We have mentioned that the answer to the question is "no" if $\kappa = \aleph_0$. We shall demonstrate that the answer to the question is also "no" for $\kappa = \aleph_1$ but is independent of the usual axioms of set theory for $\kappa \geqq \aleph_2$ (relative to the consistency of the existence of a weakly compact cardinal).

For the case $\kappa = \aleph_1$, we present this simple example. Let $X = \omega_1 \cup \{p\}$ be the one point compactification of ω_1 with the discrete topology. Let each X_α be the subspace $\alpha \cup \{p\}$. It is not hard to see that the fine topology is the same as the original topology on X.

THE SET-THEORETIC TRANSLATION. We follow standard practices with respect to set-theoretic notation. ${}^\alpha\omega$ is the set of all functions from the ordinal α into the ordinal ω. For two functions f and g in this set we write $f \leqq g$ to mean that $f(\beta) \leqq g(\beta)$ for each $\beta \in \alpha$. If $F \subset {}^\alpha\omega$ and $G \subset {}^\beta\omega$ we say that G dominates F to mean that, for each $f \in F$, there is some $g \in G$ such that $(f \mid \gamma) \leqq (g \mid \gamma)$ where γ is the minimum of α and β. When we say that G dominates f, we mean that G dominates $\{f\}$.

DEFINITION 3. For any ordinal number κ call $\{f_{\alpha,n} : \alpha \in \kappa, n \in \omega\}$ a *κ-matrix* if each $f_{\alpha,n} \in {}^\alpha\omega$. In addition, we say that it is a *coherent κ-matrix* if for each $\beta < \alpha < \kappa$ $\sup\{f_{\alpha,n}(\beta) : n \in \omega\} = \omega$ and each pair of colmns dominates each other. That is, for all α and β, $\{f_{\alpha,n} : = n \in \omega\}$ dominates $\{f_{\beta,n} : n \in \omega\}$.

We say that a κ-matrix M *extends* to a $\kappa + 1$-matrix N if for each $\alpha \in \kappa$ and $n \in \omega$, M and N have the same α, n entry.

We can now transform our question onto one about coherent matrices of functions.

THEOREM 4. *Let κ be a fixed uncountable regular cardinal. The following are equivalent.*

(a) *Any space X which has the fine topology from an increasing κ-chain of first countable spaces is also first countable.*
(b) *Any Hausdorff space X with one non-isolated point which has the fine topology from an increasing κ-chain of first countable spaces is also first countable.*
(c) *Every coherent κ-matrix can be extended to a coherent $\kappa + 1$-matrix.*
(d) *For each coherent κ-matrix there is a countable $A \subset {}^\kappa\omega$ such that every*

$g \in {}^{\kappa}\omega$ which is dominated by each column of the matrix is also dominated by A.

Before embarking on a proof, let us first examine these four conditions. The fine topology of a union of a chain of regular spaces, as in (a), may even fail to be Hausdorff; and so we could leave the realm of set-theoretic topology. However, the equivalence with (b) shows that we can just consider these zero-dimensional spaces.

Each coherent κ-matrix M gives rise to a collection of functions in ${}^{\kappa}\omega$, namely those which are dominated by each column of the matrix M. From now on let's call this collection $F(M)$. Now, (d) says that $F(M)$ has "outer" cofinality $\aleph_0$ in the $\leqq$ order on ${}^{\kappa}\omega$; while (c) says that a countable cofinal set can actually be found within $F(M)$ itself.

PROOF. (a) implies (b) is trivial; (c) implies (d) is easy since κ is uncountable. We prove that (b) implies (c) and then prove that (d) implies (a).

Assume (b), and let $M = \{f_{\alpha,n} : \alpha \in \kappa, n \in \omega\}$ be a coherent κ-matrix. We will define an increasing κ-chain of spaces which will allow (b) to tell us how to extend M. Pick $p \notin \kappa \times \omega$ and, each $\alpha \in \kappa$, let $X_\alpha = \{p\} \cup (\alpha \times \omega)$. Let τ_α be the topology on X_α generated by $\mathscr{P}(\alpha \times \omega)$, the power set of $\alpha \times \omega$, and the following neighbourhoods of p:

$$U_{\alpha,n} = \{\langle \beta, k \rangle : f_{\alpha,n}(\beta) \leqq k\} \cup \{p\} \qquad \text{for each } n \in \omega.$$

Each $\langle X_\alpha, \tau_\alpha \}$ is clearly first countable. The coherence of M shows that we have constructed an increasing κ-chain whose union with the fine topology is a Hausdorff space with one non-isolated point. Let $\{V_n : n \in \omega\}$ be a local base for p in X and for each $n \in \omega$, let

$$f_{\kappa,n}(\beta) = \min\{k : \{\beta\} \times (\omega - k)\} \subset V_n.$$

For each $\beta \in \kappa$ we have such a k because $V_n \cap X_{\beta+1}$ is open in $X_{\beta+1}$. We wish to show that $M' = M \cup \{f_{\kappa,n} : n \in \omega\}$ is coherent. The reader can check that the fact that each $X_\alpha \cap V_m$ is open in X_α gives that column α dominates column κ. The fact that $\{V_n : n \in \omega\}$ forms a local base at p shows that column κ dominates each column α.

Now assume (d) and let $X = \bigcup_{\alpha<\kappa} X_\alpha$ be the fine union of an increasing κ-chain of first countable spaces. Let $p \in X$. If p has a minimal neighbourhood in each X_α, then p has a minimal neighbourhood in X and X is first countable at p; hence without loss of generality we can assume that for each $\alpha < \kappa$ there is a strictly decreasing sequence of neighbourhoods $U_{\alpha,n}$ of p in X_α such that

$\{U_{\alpha,n} : n \in \omega\}$ is a basis for p in X_α. For each α and n we define a function $f_{\alpha,n}$ with domain α such that $f_{\alpha,n}(\beta)$ is the least $k \in \omega$ such that $U_{\beta,k} \subset U_{\alpha,n}$.

We claim that $\{f_{\alpha,n} : \alpha \in \kappa, n \in \omega\}$ forms a coherent κ-matrix. This is routine to check. We now invoke (d) to obtain the countable set A of functions. To each $f \in {}^\kappa\omega$ we associate the set

$$U_f = \{x \in X : x \in U_{\alpha, f(\alpha)}\} \qquad \text{for cofinally many } \alpha \in \kappa.$$

Clearly $p \in U_f$ and for any neighbourhood V of p in X there is an $f \in {}^\kappa\omega$ such that $U_f \subset V$. Thus in order to prove (a) it suffices to show that U_f is open. We must show that for each $\alpha \in \kappa$ the set $X_\alpha - U_f$ is closed in X_α. Let Y be a countable subset of $X_\alpha - U_f$; since X_α is first countable, it will suffice to show that no point in $X_\alpha \cap U_f$ is a limit point of Y. Choose $\beta < \kappa$ large enough so that $Y \cap U_{\gamma, f(\gamma)} = \emptyset$ for all $\gamma > \beta$. Now for each $x \in X_\alpha \cap U_f$ there is a $\gamma > \beta$ so that $x \in U_{\gamma, f(\gamma)} \cap X_\alpha$ which is an open neighbourhood of x in X_α missing Y. ∎

Combining the above theorem with Theorem 2 we obtain the following.

Corollary 5. *Suppose κ is a regular cardinal and M is a coherent κ-matrix which cannot be extended to a coherent $\kappa + 1$-matrix. Then there is no subset of ${}^\kappa\omega$ of size less than κ which dominates $F(M)$.*

Consistency one way

Our aim now is to exhibit models of set theory in which the answer to the Main Question is negative for various cardinals $\kappa > \aleph_1$.

Definition 6. Given a cardinal κ and a set $E \subset \kappa$, $\square_\kappa(E)$ is the statement: there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa)\rangle$ such that C_α is cub in α and $\gamma \in \alpha \cap C'_\alpha$ implies $\gamma \notin E$ and $C_\gamma = \gamma \cap C_\alpha$.

It is folklore that $\square_{\omega_1}$ implies the existence of a stationary set $E \subset S^0_{\omega_2} = \{\alpha \in \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ for which $\square_{\omega_2}(E)$ holds; moreover Jensen has proven that in L for each non-weakly compact κ, there is a stationary $E \subset S^0_\kappa$ such that $\square_\kappa(E)$ holds.

Theorem 7. *If κ is an uncountable regular cardinal such that $\square_\kappa(E)$ holds for some stationary $E \subset S^0_\kappa$, then the answer to the Main Question is "no". In particular, there is a coherent κ-matrix which does not extend to a coherent $\kappa + 1$-matrix.*

It turns out that we can prove Theorem 7 by constructing a rather special kind of κ-matrix.

DEFINITION 8. Two functions $f \in {}^{\alpha}\omega$ and $g \in {}^{\beta}\omega$ are said to be *parallel* when

$$\sup\{ |f(\gamma) - g(\gamma)| : \gamma \in \alpha \cap \beta\} < \omega.$$

A family of functions is said to be a parallel family if each pair of functions in the family is parallel.

The connection with matrices of functions comes from the next lemma.

LEMMA 9. *Suppose κ is a regular uncountable cardinal. If each coherent κ-matrix can be extended to a coherent $\kappa+1$-matrix, then for each parallel family $\{f_\alpha \in {}^{\alpha}\omega : \alpha \in \kappa\}$ of functions, there is an $f \in {}^{\kappa}\omega$ which is parallel to each f_α.*

PROOF. Let M be the coherent κ-matrix obtained by defining $f_{\alpha,n}(\beta) = f_\alpha(\beta) + n$ for each $\beta < \alpha < \kappa$ and $n \in \omega$. Now suppose that $\{f_{\kappa,n} : n \in \omega\}$ extends M to a coherent $\kappa+1$-matrix. For all $\alpha \in \kappa$, there are m_α, n_α so that

$$f_{\alpha,0} \leqq f_{\kappa,n_\alpha} \leqq f_{\alpha,m_\alpha}.$$

Choose n so that $\{\alpha \in \kappa : n_\alpha = n\}$ is cofinal. It is routine to verify that $f_{\kappa,n}$ is parallel to each f_α. ■

Therefore to prove Theorem 7 it suffices to build a parallel family $\{f_\alpha \in {}^{\alpha}\omega : \alpha \in \kappa\}$ for which there is no $f \in {}^{\kappa}\omega$ parallel to each f_α.

PROOF OF THEOREM 7. Fix the sequence $\langle C_\alpha : \alpha \in \lim(\kappa)\rangle$ and E as above. Note that for $\alpha \in E$ we may redefine C_α if necessary and so assume that C_α has order type ω for all $\alpha \in E$. We define the parallel family $\{f_\alpha : \alpha \in \kappa\}$ by induction. If we have already defined f_α then we shall simply define $f_{\alpha+1}$ to be $f_\alpha \cup \langle \alpha, 0\rangle$; clearly $f_{\alpha+1}$ is parallel to f_α. Now suppose that $\alpha \in \lim(\kappa)$ and we have inductively defined the parallel family $\{f_\beta : \beta \in \alpha\}$ so that if $\gamma \in C'_\beta$ then $f_\gamma = f_\beta \upharpoonright \gamma$.

Case 1. $\alpha \in E$. Now $C_\alpha = \{\alpha_n : n \in \omega\}$ is an increasing ω-sequence which is cofinal in α and we may assume that $\alpha_0 = 0$. Define f_α so that for each $n \in \omega$ we have

$$f_\alpha \upharpoonright [\alpha_n, \alpha_{n+1}) = f_{\alpha_{n+1}} + n.$$

Clearly (by induction) f_α is parallel to each f_{α_n}.

Case 2. $\alpha \notin E$ and C'_α is cofinal in α. By our inductive assumption $f_\alpha = \bigcup_{\gamma \in C_\alpha} f_\gamma$ is a function and clearly satisfies the inductive assumption.

Case 3. $\alpha \notin E$ and $C_\alpha = C_{\alpha_0} \cup \{\alpha_n : n \in \omega\}$ with $\{\alpha_n : n \in \omega\}$ increasing cofinal in α. Put

$$f_\alpha = f_{\alpha_0} \cup \bigcup_{n \in \omega} f_{\alpha_{n+1}} \mid [\alpha_n, \alpha_{n+1}).$$

Again it is clear that f_α continues the induction.

Now suppose that $f \in {}^\kappa\omega$ is parallel to each f_α. Choose an $n \in \omega$ so that $f^{\leftarrow}(n)$ is cofinal in κ. Since E is stationary, there is an $\alpha \in E$ so that $f^{\leftarrow}(n) \cap \alpha$ is cofinal in α. However f cannot be parallel to f_α since the 'lim inf' of f_α goes to infinity. ■

HISTORICAL REMARK 10. In our first version of this paper we were only able to construct a κ-matrix which could not be extended from the hypotheses in Theorem 7. We gave some further constructions of parallel families by both forcing and diamond-like assumptions. S. Todorčević then proved the following in [8]:

If κ is not weakly compact in L then there is a family $\{f_\alpha \in {}^\alpha\omega : \alpha \in \kappa\}$ of parallel functions such that there is no $f \in {}^\kappa\omega$ parallel to each f_α.

We then noticed that a slight modification of our original proof proves Theorem 7 and we have chosen to include the shorter proof of this weaker (than Todorčevićs) theorem for the sake of completeness.

WEAKLY COMPACT κ. As we discussed above we have demonstrated that the answer to the Main Question is "no" (in L) for every regular uncountable cardinal which is not weakly compact. Even more is true; by Todorčević's result the answer is "no" for any cardinal which is not weakly compact in L. Let us demonstrate that this assumption is necessary.

The following proposition lists the only properties of supercompact and weakly compact cardinals which we shall need.

PROPOSITION 11. (i) *Suppose θ is weakly compact. For any Π^1_1-formula $\varphi(X_1, \ldots, X_n)$ and any $A_1, \ldots, A_n$ in $V_{\theta+1}$ such that $\langle V_\theta, \in, A_1, \ldots, A_n \rangle \models \varphi$ there is a strongly inaccessible $\lambda < \theta$ such that V_λ is an elementary submodel of V_θ and*

$$\langle V_\lambda, \in, A_1 \cap V_\lambda, \ldots, A_n \cap V_\lambda \rangle \models \varphi.$$

(ii) *Suppose θ is supercompact. For any $\kappa \geqq \theta$ and any Π^1_1-formula $\varphi(X_1, \ldots, X_n)$ and any $A_1, \ldots, A_n$ in $V_{\kappa+1}$ such that $\langle V_\kappa, \in, A_1, \ldots, A_n \rangle \models \varphi$*

there is a strongly inaccessible $\lambda < \theta$ and an elementary submodel M of V_κ such that $|M| < \theta$, $M \cap V_\kappa = V_\lambda$ and

$$\langle M, \in, A_1 \cap M, \ldots, A_n \cap M \rangle \models \varphi.$$

PROOF. For the proof of (i), look in [5] page 297 or [2] page 171. Part (ii) follows from [6]; our restriction to Π^1_1 formulas here is unnecessary. ■

Part (ii) shows that what weak compactness does for $\kappa = \theta$, supercompactness does for all $\kappa \geqq \theta$. The reader does not need to recall the exact definition of these cardinals but we will give the explicit definition of a Π^1_1-formula. A Π^1_1-formula φ is a formula of set theory in two types of variables, x and X; furthermore φ is of the form $\forall X_0 \psi(X_0, X_1, \ldots, X_n)$ where ψ is a formula of the usual predicate logic in the language $\{\in, X_0, \ldots, X_n\}$ where the X_i are unary predicate symbols. If M is a set and $A_1, \ldots, A_n$ are subsets of M we write $\langle M, \in, A_1, \ldots, A_n \rangle \models \varphi$ if we have $\psi^M(A_0, \ldots, A_n)$ for all $A_0 \subseteq M$.

THEOREM 12. *Suppose that κ is an uncountable weakly compact cardinal. The answer to the Main Question is* "yes" *for κ. If κ is supercompact then the answer to the Main Question is "yes" for all uncountable regular $\theta \geqq \kappa$.*

PROOF. We shall just prove the theorem for the case that κ is weakly compact. By Theorem 4 it suffices to show that every coherent κ-matrix can be extended. The κ-matrix can be viewed as a function $F: [\kappa]^2 \times \omega \to \omega$ where $F(\alpha, \beta, n) = f_{\alpha,n}(\beta)$ for $\alpha > \beta$ and $n \in \omega$. We can express that the matrix cannot be extended by

$$(\forall H \in {}^{\kappa \times \omega}\omega)(\exists \alpha \in \kappa)$$
$$[(\exists m < \omega)(\forall n < \omega)(\exists \beta < \alpha)(F(\alpha, \beta, n) < H(\mu, m))$$
$$\vee (\exists n < \omega)(\forall m < \omega)(\exists \beta < \alpha)(F(\alpha, \beta, n) > H(\beta, m))].$$

Therefore the assertion that F cannot be extended is a Π^1_1-formula and we obtain a $\lambda < \kappa$ such that (essentially) $V_{\lambda+1} \models F \mid [\lambda]^2 \times \omega$ cannot be extended. From this it clearly follows that $F \mid [\lambda]^2 \times \omega$ cannot be extended. This, however, contradicts the fact that F itself extends its restriction. ■

Consistency the other way

In this section we demonstrate the relative consistency of a "yes" answer to the Main Question for $\kappa \geqq \aleph_2$. We use the equivalent (c) of Theorem 4. The general technique we follow is fairly standard. We employ a forcing iteration P_θ

of length θ where θ is, for example, a supercompact cardinal and P_θ is either the Lévy or the Mitchell collapse of θ to $\aleph_2$. We proceed indirectly. Suppose $V[G] \models M$ is a coherent κ-matrix which cannot be extended and $\kappa \geqq \aleph_2$. We fix a name $\dot{F} \subset [\kappa]^2 \times \omega \times \omega \times P_\theta$ for M. Since the forcing iteration has the θ chain condition, it is straightforward to check, similar to Theorem 12, that the corresponding forcing statement is a Π^1_1-statement over V_κ. Therefore we obtain, by Proposition 11, that there is some strongly inaccessible $\lambda < \theta$ and some $I \in [\kappa]^{<\kappa}$ so that

$$\Vdash_{P_\lambda} \dot{F} \mid [I]^2 \times \omega \text{ is a coherent matrix which cannot be extended.}$$

Therefore if we let $G_\lambda = G \cap P_\lambda$ and choose $M' \subset M$ corresponding to I we have that

$$V[G_\lambda] \models M' \text{ is a coherent matrix which cannot be extended.}$$

But now we know that M' *can* be extended in $V[G]$, hence we obtain our contradiction by proving what are known as "preservation lemmas". That is, it remains only to show that forcing with the "tail" of the iteration (i.e. P_θ / P_λ) will not introduce an extension to a coherent matrix from the ground model ($V[G_\lambda]$) which could not be extended in the ground model. Of course, for the forcings mentioned above the tail of the forcing is isomorphic to the entire itertion.

The definition of *the Lévy collapse of θ with countable conditions*, $\mathrm{Lv}(\theta, \omega_1)$, can be found in [5] along with the well-known proofs of the following proposition.

Proposition 13. *Suppose θ is strongly inaccessible.*

(i) $\mathrm{Lv}(\theta, \omega_1)$ *is ω_1-closed and has the θ-c.c.*

(ii) *If $\lambda < \theta$, then* $\mathrm{Lv}(\theta, \omega_1) = \mathrm{Lv}(\lambda, \omega_1) * \dot{Q}$ *such that if G_λ is* $\mathrm{Lv}(\lambda, \omega_1)$*-generic, then*

$$V[G_\lambda] \models \dot{Q} \cong \mathrm{Lv}(\theta, \omega_1).$$

(iii) *If G is* $\mathrm{Lv}(\theta, \omega_1)$*-generic then*

$$V[G] \models 2^{\aleph_0} = \aleph_1 \quad \textit{and} \quad 2^{\aleph_1} = \aleph_2 = \theta \quad \textit{and} \quad \aleph_1 = \aleph_1^V.$$

The Mitchell collapse $\mathrm{Mi}(\theta)$ of θ is defined in [7] and the following proposition is proved therein. However, part (i) is not explicitly stated there.

Proposition 14. *Suppose that θ is strongly inaccessible.*

(i) *There is an $\dot{R}_\theta$ such that* $\mathrm{Mi}(\theta) * \dot{R}_\theta$ *is forcing equivalent to* $\mathrm{Fn}(\theta, 2) \times Q_\theta$

where Q_θ *is* ω_1*-closed and* $\mathrm{Fn}(\theta, 2)$ *is the usual poset for adding Cohen reals.*

(ii) *If* $\lambda < \theta$ *is a limit ordinal, then* $\mathrm{Mi}(\theta) = \mathrm{Mi}(\lambda) * \dot{Q}$ *where, for any* $\mathrm{Mi}(\lambda)$*-generic* G_λ,

$$V[G_\lambda] \models \dot{Q} \cong \mathrm{Mi}(\theta).$$

(iii) $\mathrm{Mi}(\theta)$ *has the* θ-c.c.

(iv) *If* G *is* $\mathrm{Mi}(\theta)$*-generic, then*

$$V[G] \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 = \theta \quad \textit{and} \quad \aleph_1 = \aleph_1^V.$$

we now prove the preservation lemmas.

PROPOSITION 15. *Suppose that* P *is either* $\mathrm{Fn}(\theta, 2)$ *or is an* ω_1*-closed poset and suppose that* G *is* P*-generic over* V. *Suppose*

$$V \models \kappa \geqq \aleph_1 \textit{ is regular and } M = \{ f_{\alpha,n} : \alpha < \kappa, n < \omega \} \textit{ is a coherent } \kappa\textit{-matrix}.$$

Then $V[G] \models F(M) \cap V$ *is cofinal in* $F(M)$.

Recall that the definition of $F(M)$ was given after the statement of Theorem 4.

PROOF. Suppose $p \in P$ and $p \Vdash h \in F(M)$. First suppose that for some $p' < p$, for each $\alpha \in \kappa$, there is an m_α such that $p' \Vdash h \leqq f_{\alpha,m_\alpha}$. Define, in this case, $g \in {}^\kappa\omega$ by $g(\beta) = \min\{ f_{\alpha,m_\alpha}(\beta) : p' \Vdash h \leqq f_{\alpha,m_\alpha}\}$. It is easy to check that $p' \Vdash h \leqq g$ and $g \in V \cap F(M)$.

Therefore we may suppose that for each $p' < p$ there is an $\alpha < \kappa$ such that, for any $m \in \omega$, p' does not force $h \leqq f_{\alpha,m}$. We can then define $p_\varnothing = p$ and p_s for each $s \in {}^{<\omega}\omega$ by recursion as follows. Let α_s be defined as the minimum of

$$\{\alpha < \kappa : \forall m \in \omega \neg p_s \Vdash h \leqq f_{\alpha,m}\}.$$

In case P is $\mathrm{Fn}(\theta, 2)$, define $\{ p_{s^\frown n} : n \in \omega\}$ to be a maximal-below-p_s antichain so that there is an indexing of ω $\{k_n : n \in \omega\}$ so that $p_{s^\frown n} \Vdash h \nleqq f_{\alpha_s,k_n}$. In case P is ω_1-closed we do not require that $\{ p_{s^\frown n} : n \in \omega\}$ is maximal below p_s hence we may then assume that $k_n = n$.

Now let $\alpha = \sup\{\alpha_s : s \in {}^{<\omega}\omega\}$ and note that $\delta < \kappa$.

Let us complete the case that P is ω_1-closed. For each $n \in \omega$ and each $s \in {}^{<\omega}\omega$ there is an integer $k(s, n)$ such that $f_{\alpha,n} \leqq f_{\alpha_s,k(s,n)}$ since the matrix is coherent. Therefore we can choose the recursively defined branch $\{s_n : n \in \omega\}$ where $s_{n+1} = s_n{}^\frown k(s_n, n)$. Since P is ω_1-closed we can choose p_ω below each p_{s_n}

and note that $p_\omega \Vdash h \not\leqq f_{\alpha,n}$ for each $n \in \omega$ since $p_\omega \Vdash h \not\leqq f_{\alpha_{s_n},k(s_n,n)}$) and $f_{\alpha,n} \leqq f_{\alpha_{s_n},k(s_n,n)}$. Therefore $p_\omega \Vdash \dot{h} \notin F(M)$ — a contradiction.

Now we complete the Cohen real proof. Suppose that $q < p$ and $m \in \omega$ are such that $q \Vdash \dot{h} \leqq f_{\alpha,m}$ (there must be such a q since p forces $\dot{h}$ is in $F(M)$). It can then be proved that there must be $s \in {}^{<\omega}\omega$ so that $p_{s^\frown n}$ is compatible with q for each $n \in \omega$. With such an s, we can now choose n large enough so that $f_{\alpha,m} \leqq f_{\alpha_s,n}$ and obtain a contradiction from the fact that $p_{s^\frown n} \Vdash \dot{h} \leqq f_{\alpha,m}$. ∎

COROLLARY 16. *Suppose G is* Mi(θ)*-generic over V for some cardinal θ. Suppose further that $V \vDash \kappa \geqq \aleph_1$ is a regular cardinal and $M = \{f_{\alpha,n} : \alpha < \kappa, n \in \omega\}$ is a coherent κ-matrix. Then $V[G] \vDash F(M) \cap V$ is cofinal in $F(M)$.*

PROOF. Choose $\dot{R}_\theta$ as in Proposition 14 so that $\mathrm{Mi}(\theta) * \dot{R}_\theta$ is forcing isomorphic to $Q_\theta \times \mathrm{Fn}(\theta, 2)$ where Q_θ is ω_1-closed. If H is $\dot{R}_\theta$-generic over $V[G]$ then, by Proposition 15,

$$V[G * H] \vDash F(M) \cap V \text{ is cofinal in } F(M)$$

since we can regard the extension as being obtained by first forcing with the ω_1-closed poset Q_θ followed by forcing with $\mathrm{Fn}(\theta, 2)$.

It of course follows that $V[G] \vDash F(M) \cap V$ is cofinal in $F(M)$. ∎

Putting all the ingredients together we finally obtain the following theorem.

THEOREM 17. *Let P be the partial orders described below and G be P-generic over V. CH denotes the continuum hypothesis.*

(i) *If θ is supercompact and P is* $\mathrm{Lv}(\theta, \omega_1)$, *then*
$V[G] \vDash$ CH holds and for each regular $\kappa \geqq \aleph_2$, the answer to the Main Question is "yes".

(ii) *If θ is supercompact and P is* Mi(θ), *then*
$V[G] \vDash$ CH fails and for each regular $\kappa \geqq \aleph_2$, the answer to the Main Question is "yes".

(iii) *If θ is weakly compact and P is* $\mathrm{Lv}(\theta, \omega_1)$, *then*
$V[G] \vDash$ CH holds and for $\kappa = \aleph_2$, the answer to the Main Question is "yes".

(iv) *If θ is weakly compact and P is* Mi(θ), *then*
$V[G] \vDash$ CH fails and for $\kappa = \aleph_2$, the answer to the Main Question is "yes".

REMARK 18. One can also prove an interesting modification of Theorem 17, namely force that for cofinally many κ the answer to the Main Question is "yes" and for cofinally many κ the answer is "no". One method is to start with the model produced in Theorem 17(i) and then force a "no" answer at a class of cardinals. This forcing would be an iteration of the $\square_\kappa$ forcing as in [1, Theorem 24]. This iteration would have an Easton-type support and Lemma 15 guarantees that "no" answers are preserved by the tail of the forcing.

An alternate method only requires that there are arbitrarily large weakly compact cardinals rather than a supercompact cardinal. The consistency of this follows from that of a measurable cardinal. However, with this method, we may well be stuck with a "no" answer at κ^+ for all singular κ. One uses L as the ground model and Lévy collapses (with Easton support) the next weakly compact to the successor (or the second successor etc.) of each weakly compact cardinal. In the resulting model, the answer is "yes" at κ iff κ is weakly compact in L iff κ is the successor of an uncountable regular cardinal.

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